A Timoshenko beam theory with pressure corrections for plane stress problems

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Abstract

A Timoshenko beam theory for plane stress problems is presented. The theory consists of a novel combination of three key components: average displacement and rotation variables that provide the kinematic description of the beam, stress and strain moments used to represent the average stress and strain state in the beam, and the use of exact axially-invariant plane stress solutions to calibrate the relationships between all these quantities. These axially-invariant solutions — called the fundamental states — are also used to determine a shear strain correction factor as well as corrections to account for effects produced by externally applied loads. The shear strain correction factor and the external load corrections are computed for a beam composed of isotropic layers. The proposed theory yields Cowper's shear correction for a single isotropic layer, while for multiple layers new expressions for the shear correction factor are obtained. A body-force correction is shown to account for the difference between Cowper's shear correction and the factor originally proposed by Timoshenko. Numerical comparisons between the theory and finite-elements results show good agreement.

Keywords: Timoshenko beam theory, shear correction factor

1. Introduction

The equations of motion for a deep beam that include the effects of shear deformation and rotary inertia were first derived in two papers by Timoshenko (1921, 1922). Two essential aspects of Timoshenko's beam theory are the treatment of shear deformation by the introduction of a mid-plane rotation variable and the use of a shear correction factor. The definition and value of the shear correction factor have been the subject of numerous research papers, a few of which are discussed below. Timoshenko's approach of using the

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mid-surface displacement and mid-surface rotation variables has been presented by many different authors. Shames and Dym (1985, Ch. 4, pg. 197) provide an excellent overview of this approach. In this paper, we concentrate on theories that refine the approximations Timoshenko used in his original paper.

Prescott (1942), derived the equations of vibration for thin rods using average through-thickness displacement and average rotation variables. He introduced a shear correction factor to account for the difference between the average shear on a cross section and the expected quadratic distribution of shear.

Cowper (1966) presented a revised derivation of Timoshenko's beam theory starting from the equations of elasticity for a linear, isotropic beam in static equilibrium. Cowper introduced residual displacement terms that he defined as the difference between the actual displacement in the beam and the average displacement representation. These residual displacements account for the difference between the average shear strain and the shear strain distribution. Cowper introduced a correction factor to account for this difference and computed its value based on three-dimensional solutions of cantilever beams with various cross sections subjected to a tip load.

Stephen and Levinson (1979), developed a beam theory along the lines of Cowper's but recognized that the variation in shear along the length of the beam would lead to a modification of the relationship between bending moment and rotation. This variation was neglected by Cowper.

In this paper, we present a beam theory that follows the work of Cowper (1966) and Stephen and Levinson (1979). Similar to these authors, we seek a solution to a beam problem based on average through-thickness displacement and rotation variables. In a departure from previous work, we introduce strain moments that are analogous to the stress moments used in the equilibrium equations. These strain moments remove the restriction of working with an isotropic, homogeneous beam. This is an essential component of the present approach, as sandwich and layered orthotropic beams are used for many high-performance, aerospace applications (Flower and Soutis, 2003).

Another important feature of the theory is the use of certain statically determinate beam problems that we use to construct the relationship between stress and strain moments, and to reconstruct the stress and strain solution in a post-processing step. We call these solutions the fundamental states of the beam. These ideas were first pursued by Hansen and Almeida (2001) and Hansen et al. (2005) and an extension of this theory to the analysis of plates is presented by Guiamatsia and Hansen (2004), Tafeuvoukeng (2007) and Guiamatsia (2010).

The paper begins with a brief discussion of two classical methods used to calculate the shear correction factor. Section (3) describes the proposed theory and section (3.1) introduces the fundamental states. In section (4) calculations are presented for a beam composed of multiple isotropic layers. Section (5) briefly presents the modified equations of motion for an isotropic beam. In section (6) comparisons are made with finite-element calculations.

2. The shear correction factor

One of the main difficulties in using Timoshenko beam theory is the proper selection of the shear correction factor. Many authors have published definitions of the shear correction factor and have proposed various methods to calculate it. Most of these approaches fall into one of two categories. The first approach is to use the shear correction factor to match the frequencies of vibration of various beam constructions with exact solutions to the theory of elasticity. The second approach is to use the shear correction factor to account for the difference between the average shear or shear strain and the actual shear or shear strain using exact solutions to the theory of elasticity.

Timoshenko (1922) originated the frequency-matching approach. He calculated the shear correction factor by equating the frequency of vibration determined using the plane stress equations of elasticity to those computed using his beam theory. Although not explicitly written in the paper, the shear correction factor obtained in this manner for a rectangular beam is,

$$k_{xy} = \frac{5(1+\nu)}{6+5\nu}. \tag{1}$$

Cowper (1966) calculated the shear correction factor based on a different approach. Using residual displacements, designed to take into account the distortion of the cross sections under shear loads, Cowper was able to derive a formula for the shear correction factor based on solutions of a cantilever beam subjected to a tip load. For a rectangular isotropic homogeneous beam, Cowper found a shear correction factor of,

$$k_{xy} = \frac{10(1+\nu)}{12+11\nu}. \tag{2}$$

Stephen (1980) computed the shear correction factor for beams of various cross sections by using the exact solutions for a beam subject to a uniform gravity load. He employed a modified form of the Kennard-Leibowitz method (Leibowitz and Kennard, 1961), to obtain the shear correction factor by equating the average center-line curvature of the exact result with the Timoshenko solution. He obtained a modified form of Timoshenko's shear correction factor for rectangular sections that approached equation (1) for thin cross-sections.

Hutchinson (1981) computed the shear correction factor by performing a comparison between Timoshenko beam theory and three solutions from the theory of elasticity, the Pochhammer–Chree solution in **?**, a Fourier solution due to Pickett (1944) and a series solution computed by Hutchinson (1980). Hutchinson found that the best shear correction factor was dependent on the frequency and Poisson's ratio of the beam, but that Timoshenko's value was better than Cowper's.

In another paper, Hutchinson (2001) introduced a new Timoshenko beam formulation and computed the shear correction factor for various cross sections based on the a comparison with a tip-loaded cantilever beam. For a beam with a rectangular cross section, Hutchinson obtained a shear correction factor that depends on the Poisson ratio and the width to depth ratio. In a later discussion of the paper, Stephen (2001) showed that the values he obtained in (Stephen, 1980) were equivalent.

More recently Dong et al. (2010), presented a semi-analytic finite-element technique for calculating the shear correction factor based either on the Saint–Venant warping function or the free vibration of a beam.

Some experimental studies have been performed to try and measure the shear correction factor based on the original equations proposed by Timoshenko. Spence and Seldin (1970) obtained experimental values of the shear correction factor for a series of square and circular beams composed of both isotropic and anisotropic materials by determining their natural frequencies. Kaneko (1975) performed an extensive review of the shear correction factors for rectangular and circular cross sections obtained by various authors using either experimental techniques or analysis. These experimental studies have generally used a natural frequency approach to determining the shear correction factor and have generally found that Timoshenko's value is superior to Cowper's. This perhaps, is not surprising since Timoshenko's correction is obtained by matching frequencies in the same manner in which the experiments are performed. However, these methods fail to provide a theoretical explanation as to why the value of a factor that modifies the relationship between the shear resultant and the average shear strain should be determined by the natural frequency of vibration. It is this deficiency that motivates the following work.

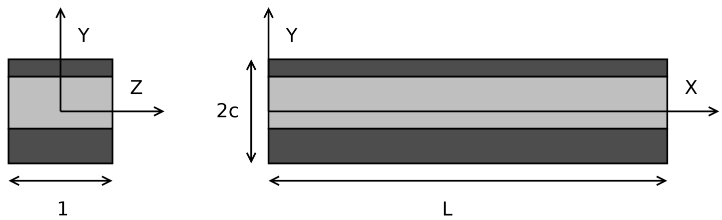

Figure 1: The geometry of the beam

3. The theory

The geometry of the beam under consideration is shown in Figure (1). The beam extends along the x-direction subject to forces on the top and bottom surfaces in the y-direction. The reference axis is placed at the centroid of the cross-section of the beam. The beam is of uniform composition in both the x and z-directions and so consists of a series of layers with different material properties. We assume that each layer is composed of an isotropic material with different material properties. These assumptions eliminate the possibility of twisting and allow the beam to be modeled using a plane stress assumption in the z plane. Although variation in the Poisson's ratio between layers would lead to a violation of the plane stress assumption, we include this possibility and ignore the edge effects in such situations. The half-thickness of the beam in the y-direction is c while the length of the beam in the x-direction is L.

Following Prescott (1942) and Cowper (1966), the average, through thickness displacements and an average rotation are defined as follows,

$$\begin{aligned} u_0(x,t) &= \frac{1}{2c}\int_{-c}^{c} u(x,y,t)\,dy, \\ u_1(x,t) &= \frac{3}{2c^3}\int_{-c}^{c} yu(x,y,t)\,dy, \\ v_0(x,t) &= \frac{1}{2c}\int_{-c}^{c} u(x,y,t)\,dy, \end{aligned} \tag{3}$$

where u and v are the displacements in the x and y directions respectively. The average displacements and rotation are defined regardless of the through-thickness behaviour of u and v that are piecewise continuous through the thickness of the beam in this problem. The average displacements are an incomplete representation of the total displacement field in the beam in the sense that the average quantities do not capture the exact pointwise displacements everywhere. In order to capture these distortion effects, it is necessary to introduce residual displacements that account for the difference between the average and pointwise quantities in the following manner,

$$\begin{aligned} u(x,y,t) &= u_0(x,t) + yu_1(x,t) + \tilde{u}(x,y,t), \\ v(x,y,t) &= v_0(x,t) + \tilde{v}(x,y,t), \end{aligned} \tag{4}$$

where $\tilde{u}(x,y)$ and $\tilde{v}(x,y)$ are the residual displacements in the x and y directions as introduced by Cowper (1966). From the definitions of the average displacements (3), the zeroth and first moment of $\tilde{u}$ and the zeroth moment of $\tilde{v}$ through the thickness are zero,

$$\begin{aligned} \int_{-c}^{c} \tilde{u}(x,y,t)\, dy &= 0, \\ \int_{-c}^{c} y\tilde{u}(x,y,t)\, dy &= 0, \\ \int_{-c}^{c} \tilde{v}(x,y,t)\, dy &= 0. \end{aligned}$$

The average displacements and displacement residuals may be used to determine the strain at any point in the beam. In the approach that follows however, we are interested not in the pointwise distribution of the strain, but in the average strain in the through-thickness direction. To this end, we introduce the following strain moments,

$$\epsilon_0' = \int_{-c}^{c} \frac{\partial u}{\partial x}\, dy = 2c\frac{\partial u_0}{\partial x}, \tag{5a}$$

$$\kappa' = \int_{-c}^{c} y\frac{\partial u}{\partial x}\, dy = \frac{2c^3}{3}\frac{\partial u_1}{\partial x}, \tag{5b}$$

$$\gamma' = \int_{-c}^{c} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\, dy = 2c\left[u_1 + \frac{\partial v_0}{\partial x}\right] + \int_{-c}^{c} \frac{\partial \tilde{u}}{\partial y}\, dy. \tag{5c}$$

These variables are analogous to the stress moments that are used to define the equilibrium equations for a beam. The primes are used here to denote the total strain moment as it will be important to distinguish between different contributions to the these quantities. Note that these strain moments are not normalized and as a result have different dimensions than the pointwise strain.

Thus far, no assumptions beyond those of linear elasticity have been made. The combination of the average and residual displacements can be used to capture any arbitrary displacement field no matter what the applied loads.

3.1. The fundamental states

The basic assumption made in the development of this beam theory is that the stress and strain state is well approximated by a linear combination of axially-invariant solutions. These solutions come from a set of specially chosen statically determinate beam problems that we refer to as the fundamental states. The first three fundamental states play an important role in the theory and are used to construct a constitutive relationship between the stress and strain moments. These first three fundamental states are solutions corresponding to an axial load, a constant bending moment and a constant shear load in a beam with identical construction to the beam under consideration. The solutions are normalized such that the applied loads are of unit magnitude.

Fundamental states are also associated with external loads applied to the beam. The magnitudes of these fundamental states are known from the loading conditions. Since the through-thickness stress and strain distributions may alter the stress or strain moments, a distinction must be made between the way loads are applied to the beam. The strain moments resulting from pressure loads or body loads may be different, even if the magnitude of the load is identical.

Using the assumption that the stress and strain state in the beam is a linear combination of the fundamental states, the stress and strain distribution in the beam may be written as follows,

$$\sigma(x,y,t) = N\sigma^N + M\sigma^M + Q\sigma^Q + P\sigma^P, \tag{6a}$$

$$\epsilon(x,y,t) = N\epsilon^N + M\epsilon^M + Q\epsilon^Q + P\epsilon^P, \tag{6b}$$

where the superscripts denote the appropriate fundamental states. Here P may stand for any externally applied load while N, M and Q are the axial resultant, bending moment and shear resultant respectively. The fundamental states are only functions of the through-thickness position y, while the stress resultants and loads are functions of axial location x and time t. In order to retain the proper relationship between the stress moments and the stress resultants, the pressure stress state σ^P must not contribute to the axial resultant, bending moment or shear resultant. There are no such restrictions on the fundamental states for strain.

Using Equation (6b), the zeroth moment of the axial strain is,

$$\epsilon_0' = N\epsilon_0^N + M\epsilon_0^M + Q\epsilon_0^Q + P\epsilon_0^P,$$

where again, the superscripts denote the fundamental state used to evaluate the appropriate strain moment. The relationships for the remaining strain moments are analogous.

Here we distinguish between the known and unknown contributions to the strain moments. The unknown contributions come from the first three fundamental states and are denoted without primes. The known contributions arise from external loads and are denoted by an over-bar, thus the total contribution to the zeroth axial strain moment is,

$$\epsilon_0' = \epsilon_0 + \bar{\epsilon}_0,$$

where $\bar{\epsilon}_0 = P\epsilon_0^P$.

It is important to emphasize that the fundamental state solutions are independent of end conditions. Average stress resultant conditions are imposed at the ends of the beam such that the axial, bending and shear resultants are in equilibrium with any applied loads. Rigid body translation and rotation are removed from the solution by a set of displacement constraints.

The relationship between the axial force, bending moment and shear resultants and the equivalent strain moments is determined using the first three fundamental states.

This linear relationship is,

$$\begin{bmatrix} N \\ M \\ Q \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} \begin{bmatrix} \epsilon_0 \\ \kappa \\ \gamma \end{bmatrix}. \tag{7}$$

The constitutive matrix is determined using the magnitudes of the strain moments in each of the first three fundamental states as follows,

$$\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_0^N & \epsilon_0^M & \epsilon_0^Q \\ \kappa^N & \epsilon_0^M & \kappa^Q \\ \gamma^N & \gamma^M & \gamma^Q \end{bmatrix}^{-1}. \tag{8}$$

Due to the construction of the beam, the direct and shear stresses do not couple and so $\gamma^N = \gamma^M = \epsilon_0^Q = \kappa^Q = 0$. Furthermore, if the beam is homogeneous and isotropic $D_{11} = D_{22} = E$ with $D_{12} = D_{21} = 0$ and $D_{33} = G$.

3.2. The shear strain correction

The additional integral in the expression for the shear strain moment from Equation (5c) involves a correction from the residual displacements. The value of this integral depends on the distribution of the shear strain through the thickness. Several authors have suggested that this shear strain correction should be computed under different loading conditions. For example, Cowper (1966) computed his value of the shear correction factor for a beam subject to a constant shear load while Stephen (1980) and Hutchinson (2001) compute the correction for a beam subject to a gravity load. We take the shear strain correction to be equal to the ratio of the shear strain moment to the average shear strain computed using the fundamental state corresponding to shear,

$$k_{xy} = \frac{\gamma^Q}{2c\left[u_1 + \frac{\partial v_0}{\partial x}\right]_Q} = 1 + \frac{\left.\int_{-c}^{c} \frac{\partial \tilde{u}}{\partial y}\, dy\right|_Q}{2c\left[u_1 + \frac{\partial v_0}{\partial x}\right]_Q}. \tag{9}$$

The corrected shear strain moment is thus,

$$\gamma = 2ck_{xy}\left[u_1 + \frac{\partial v_0}{\partial x}\right].$$

It is important to note that this correction is not a correction on the shear stiffness of the beam, but rather a correction on the discrepancy between the average shear strain and the displacement representation. It is therefore, more correct to refer to it as a shear strain correction.

3.3. The pressure correction

When pressure loads are applied to the beam, the relationship between the strain and stress moments expressed by Equation (7), is no longer valid since it was produced under the assumption that no external loads are applied to the beam. The proper relationship to use between the strain and stress moments is,

$$\epsilon_0 = 2c\frac{\partial u_0}{\partial x} - P\epsilon_0^P, \tag{10a}$$

$$\kappa = \frac{2c^3}{3}\frac{\partial u_1}{\partial x} - P\kappa^P, \tag{10b}$$

$$\gamma = 2ck_{xy}\left[u_1 + \frac{\partial v_0}{\partial x}\right] - P\gamma^P, \tag{10c}$$

where here ϵ_0, κ and γ are the contributions to the strain moment from the first fundamental state. These strain moments result in a correction to the stress moments as follows,

$$\begin{bmatrix} N \\ M \\ Q \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} \begin{bmatrix} \epsilon_0' \\ \kappa' \\ \gamma' \end{bmatrix} - P \begin{bmatrix} N^P \\ M^P \\ Q^P \end{bmatrix}, \tag{11}$$

where N^P, M^P and Q^P are the product of the strain moments, ϵ_0^P, κ^P and γ^P and the averaged constitutive relation (7). This modified constitutive relationship must be used when external loads are applied to the beam.

3.4. Equilibrium equations

The equilibrium equations for the stress resultants are obtained by the standard approach of integrating the two-dimensional, stress-equilibrium equations. When the density of the material is constant ρ, these equations are,

$$\frac{\partial N}{\partial x} = 2c\rho\frac{\partial^2 u_0}{\partial t^2}, \tag{12a}$$

$$\frac{\partial M}{\partial x} - Q = \frac{2c^3}{3}\rho\frac{\partial^2 u_1}{\partial t^2}, \tag{12b}$$

$$\frac{\partial Q}{\partial x} + P = 2c\rho\frac{\partial^2 v_0}{\partial t^2}. \tag{12c}$$

If the density of the material varies in the through-thickness direction, these equations would involve integrals of the residual displacements.

4. Isotropic layered beam

In this section we derive the fundamental states, the stress-strain moment constitutive equation, the shear correction factor and the pressure strain moment corrections for a beam composed of K isotropic layers. Each layer has Young's modulus E_k, Poisson's ratio ν_k, and is situated between $y = h_k$ and $y = h_{k+1}$ where h_k is defined relative to the centroid of the cross section. Define the ratio of the Young's moduli α_k where $E_k = E\alpha_k$, where E may be chosen to be equal to the Young's modulus in any convenient layer. Furthermore, define the non-dimensional ratio of the stations of the layers as follows $\xi_k = h_k/c$. For convenience in presenting various formula, we define $\Delta_k^n = h_{k+1}^n - h_k^n$

and $\delta_k^n = \xi_{k+1}^n - \xi_k^n$. Also, we define the weighted area and weighted second moment of area and a stretching-bending parameter as follows,

$$A \equiv \sum_{i=1}^{K} \alpha_i \Delta_k, \quad I \equiv \sum_{i=1}^{K} \frac{\alpha_i}{3} \Delta_k^3, \quad t \equiv \frac{1}{A} \sum_{i=1}^{K} \frac{\alpha_i}{2} \Delta_k^2.$$

In the following formula, a subscript k will be used to define the stress or strain distribution in the k-th layer, lying between $h_k \leq y \leq h_{k+1}$.

4.1. Axial and bending states

The first fundamental state solution corresponds to a beam subject to a unit axial load that results in the following stress,

$$\sigma_{x\ (k)}^{N} = \frac{I}{A} \frac{1}{I - At^2} \alpha_k \left(1 - ry\right),$$

where $r = tA/I$. The strain moments in this fundamental state are,

$$\epsilon_0^N = \frac{2cI}{A} \frac{1}{E(I - At^2)}, \quad \kappa^N = -\frac{2c^3 t}{3} \frac{1}{E(I - At^2)},$$

and $\gamma^N = 0$.

The second fundamental state solution corresponds to a unit bending moment that results in the following stress,

$$\sigma_{x\ (k)}^{M} = \frac{1}{I - At^2} \alpha_k \left(y - t\right).$$

The strain moments in this fundamental state are,

$$\epsilon_0^M = -2ct \frac{1}{E(I - At^2)}, \quad \kappa^M = \frac{2c^3}{3} \frac{1}{E(I - At^2)},$$

and $\gamma^M = 0$. Using Equation (8), the relationship between the strain moments and the stress moments, can be determined as follows,

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = E(I - At^2) \begin{bmatrix} 2cI/A & -2ct \\ -2c^3 t/3 & 2c^3/3 \end{bmatrix}^{-1} = \frac{3EA}{4c^4} \begin{bmatrix} 2c^3/3 & 2ct \\ 2c^3 t/3 & 2cI/A \end{bmatrix},$$

4.2. Shear state and shear strain correction

The third fundamental state corresponds to a constant unit shear load. The stresses corresponding to this case are,

$$\sigma_{(k)} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_{(k)} = \frac{1}{2(I - At^2)} \begin{bmatrix} 2\alpha_k x(y - t) \\ 0 \\ \alpha_k (c_k + 2ty - y^2) \end{bmatrix}, \tag{13}$$

where the c_i terms are determined to ensure a continuous variation of the shear stress through the thickness and are determined through the following equations, $c_1 = h_1^2 - 2th_1$ and $c_i = \left((\alpha_{i-1} - \alpha_i)(2th_i - h_i^2) + \alpha_{i-1} c_{i-1}\right)/\alpha_i$.

The fundamental state consists of only the stresses corresponding to the axial-invariant components of the solution. These are obtained by setting $\sigma^Q(y) = \sigma(x = 0, y)$ from Equation (13).

The shear strain moment is determined by integrating the shear strain through the thickness,

$$\gamma^Q = \sum_{k=1}^{K} \frac{(1+\nu_k)}{E(I - At^2)} \left(c_k \Delta_k + t\Delta_k^2 - \frac{1}{3}\Delta_k^3 \right).$$

The relationship between the shear stress resultant and the shear strain resultant is, $Q = D_{33}\gamma$ where,

$$D_{33} = 1/\gamma^Q. \tag{14}$$

This is not a simple average of the shear-modulus through the thickness that is often used in beam theories. Equation (14) is a weighted average dependent on the relative distribution of shear through the thickness.

The shear correction factor for the multi-layer beam is determined using Equation (9). It is a dimensionless quantity that depends only on the relative position of the layers and the relative magnitudes of the stiffnesses of each layer. As such, it is expressed using dimensionless quantities.

The dimensionless bending-stretching coupling constant is given by τ,

$$\tau = \frac{1}{2}\sum_{k=1}^{K} \alpha_k \left(\xi_{k+1}^2 - \xi_k^2\right) \Bigg/ \sum_{k=1}^{K} \alpha_k \left(\xi_{k+1} - \xi_k\right).$$

We next introduce the constants C_k, B_k and A_k that are defined sequentially for each layer. For $k = 1$, $C_1 = \xi_1^2 - 2\tau\xi_1$, $B_1 = -2(1+\nu_1)C_1$ and $A_1 = 0$. For each other layer,

$$\begin{aligned}
C_k &= \left((\alpha_{k-1} - \alpha_k)(2\tau\xi_k - \xi_k^2) + \alpha_{k-1}C_{k-1}\right)/\alpha_k, \\
B_k &= (\nu_{k-1} - \nu_k)(\xi_k^2 - 2\tau\xi_k) + B_{k-1}, \\
A_k &= 2\xi_k \left((1+\nu_{k-1})C_{k-1} - (1+\nu_k)C_k + \frac{1}{2}(B_{k-1} - B_k) \right) \\
&\quad + (\nu_{k-1} - \nu_k)(\tau\xi_k^2 - \xi_k^3/3) + A_{k-1}.
\end{aligned}$$

The shear correction factor for the layered, isotropic beam is,

$$k_{xy} = D/F, \tag{15}$$

where,

$$\begin{aligned}
D &= \sum_{k=1}^{K} (1+\nu_k) \Big\{ C_k\delta_k + \tau\delta_k^2 - \frac{1}{3}\delta_k^3 \Big\}, \\
F &= \sum_{k=1}^{K} \Big\{ \frac{1}{2}\delta_k^3 (2(1+\nu_k)C_k + B_k) + \frac{1}{40}(2+\nu_k)\left(15\tau\delta_k^4 - 4\delta_k^5\right) \\
&\quad + \frac{3}{4}A_k\delta_k^2 - \frac{1}{2}B_k\delta_k + \frac{\nu_k}{2}\left(\tau\delta_k^2 - \frac{1}{3}\delta_k^3\right) \Big\}.
\end{aligned}$$

For a single-layer beam, this expression simplifies to Cowper's shear correction factor from Equation (2).

4.3. Pressure state and pressure strain correction

The fourth fundamental state corresponds to a pressure load applied to the beam. The total force in the y-direction per unit length of the beam is distributed between a traction on the top surface P_t and a traction on the bottom surface P_b that both act in the positive y direction. The total force is such that the contributions sum to unity $P_t + P_b = 1$.

The pressure load causes a linearly varying shear and quadratically varying moment in the beam resulting in the following state of stress,

$$\sigma_{(k)} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_{(k)} = \frac{1}{2(I - At^2)} \begin{bmatrix} -\alpha_k \left(x^2 y - tx^2 - 2y^3/3 + 2ty^2 + e_k y + f_k\right) \\ \alpha_k \left(d_k + c_k y + ty^2 - y^3/3\right) \\ -\alpha_k x \left(c_k + 2ty - y^2\right) \end{bmatrix}. \tag{16}$$

The fundamental state is determined by taking only the axially-invariant components of the stress state given in Equation (16), $\sigma^P(y) = \sigma(x = 0, y)$.

The coefficients d_k are determined from inter-layer continuity of σ_y while the coefficients e_k and f_k are used to satisfy two equilibrium equations $\int_{-c}^{c} y\sigma_x dy = -x^2/2$ and $\int_{-c}^{c} \sigma_x dy = 0$ as well as $K - 2$ inter-layer displacement continuity constraints. The d_k coefficients can be determined using the following relationship, $d_1 = -2(I - At^2)P_b/\alpha_1 - (c_1 h_1 + th_1^2 - h_1^3/3)$ and

$$d_k = 1/\alpha_k \left[(\alpha_{k-1} - \alpha_k)(th_k^2 - h_k^3/3) + h_k(\alpha_{k-1}c_{k-1} - \alpha_k c_k) + \alpha_{k-1}d_{k-1}\right].$$

The additional equations for the inter-layer continuity of the displacements are,

$$(e_k - e_{k-1})h_k - (f_k - f_{k-1}) = (\nu_{k-1} - \nu_k)\left(th_k^2 - h_k^3/3\right) - \nu_k \left(d_k + c_k h_k\right) + \nu_{k-1}\left(d_{k-1} + c_{k-1}h_k\right),$$
$$e_k - e_{k-1} = c_k(2 + \nu_k) - c_{k-1}(2 + \nu_{k-1}) + (\nu_{k-1} - \nu_k)(h_k^2 - 2th_k),$$

for $k = 2, \ldots, K$. The two additional equilibrium equations are,

$$\sum_{i=1}^{K} \alpha_k \left\{\frac{e_k}{3}\Delta_k^3 + \frac{f_k}{2}\Delta_k^2\right\} = \sum_{i=1}^{K} \alpha_k \left\{\frac{2}{15}\Delta_k^5 - \frac{t}{2}\Delta_k^4\right\},$$
$$\sum_{i=1}^{K} \alpha_k \left\{\frac{e_k}{2}\Delta_k^2 + f_k \Delta_k\right\} = \sum_{i=1}^{K} \alpha_k \left\{\frac{1}{6}\Delta_k^4 - \frac{2t}{3}\Delta_k^3\right\}.$$

Using the values obtained by solving these for e_k and f_k, with the above $2K$ equations, the strain moments for this fundamental state may be obtained,

$$\epsilon_0^P = \frac{1}{2E(I - At^2)} \left\{\frac{4}{3}tc^3 + \sum_{k=1}^{K} \left(\frac{e_k}{2}\Delta_k^2 + f_k\Delta_k + \nu_k \left(d_k\Delta_k + \frac{c_k}{2}\Delta_k^2 + \frac{t}{3}\Delta_k^3 - \frac{1}{12}\Delta_k^4\right)\right)\right\}, \tag{17a}$$

$$\kappa^P = \frac{1}{2E(I - At^2)} \left\{-\frac{4}{15}c^5 + \sum_{k=1}^{K} \left(\frac{e_k}{3}\Delta_k^3 + \frac{f_k}{2}\Delta_k^2 + \nu_k \left(\frac{d_k}{2}\Delta_k^2 + \frac{c_k}{3}\Delta_k^3 + \frac{t}{4}\Delta_k^4 - \frac{1}{15}\Delta_k^5\right)\right)\right\}. \tag{17b}$$

The shear strain moment is zero, $\gamma^P = 0$.

5. Equations of motion for an isotropic beam

Before examining several static cases using the shear and pressure corrections derived above, we will briefly examine the natural frequency of vibration of an isotropic beam with a body-force correction. For this isotropic case, $I = 2c^3/3$, $A = 2c$ and $t = 0$.

Under a constant body load with a value of $1/2c$, the stress state in an isotropic beam is,

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \frac{1}{2I} \begin{bmatrix} -x^2y + 2y^3/3 - 2c^2y/5 \\ (c^2y - y^3)/3 \\ xy^2 - xc^2 \end{bmatrix}. \tag{18}$$

The stresses have a linear varying shear and a quadratically varying bending moment, as in the pressure state described above. The fundamental state corresponding to a body load is, $\sigma^B(y) = \sigma(x = 0, y)$ from Equation (18). The strain moments corresponding to this fundamental state are, $\epsilon_0^B = 0$, $\gamma^B = 0$ and

$$\kappa^B = -\frac{1}{2EI}\frac{\nu}{3}\left(\frac{2c^5}{3} - \frac{2c^5}{5}\right) = -\frac{\nu c^2}{15E}. \tag{19}$$

The bending moment correction is $M^B = -\nu c^2/15$. Under conditions of free-vibration, the magnitude of this body-force fundamental state is equal to the inertial force per unit span. As a result, Equation (11) becomes,

$$M = EI\frac{\partial u_1}{\partial x} + \rho A M^B \frac{\partial^2 v_0}{\partial t^2}. \tag{20}$$

Using this constitutive relationship, the equation of motion for a freely vibrating beam is,

$$EI\frac{\partial^4 v_0}{\partial x^4} + \rho A\frac{\partial^2 v_0}{\partial t^2} - \rho I\left[1 + \frac{E}{k_{xy}G} + \frac{AM^B}{I}\right]\frac{\partial^4 v_0}{\partial t^2 \partial x^2} + \frac{\rho^2 I}{k_{xy}G}\frac{\partial^4 v_0}{\partial t^4} = 0. \tag{21}$$

The classical equation of motion may be obtained by setting $M^B = 0$. The equation of motion for an isotropic beam, using the body-force correction (19) and Cowper's shear correction factor (2) is,

$$EI\frac{\partial^4 v_0}{\partial x^4} + \rho A\frac{\partial^2 v_0}{\partial t^2} - \frac{17 + 10\nu}{5}\rho I\frac{\partial^4 v_0}{\partial t^2 \partial x^2} + \frac{12 + 11\nu}{5}\left(\frac{\rho I}{A}\right)^2\frac{\partial^4 v_0}{\partial t^4} = 0, \tag{22}$$

while for the classical equation, with Timoshenko's shear correction factor (1), the equation of motion is,

$$EI\frac{\partial^4 v_0}{\partial x^4} + \rho A\frac{\partial^2 v_0}{\partial t^2} - \frac{17 + 10\nu}{5}\rho I\frac{\partial^4 v_0}{\partial t^2 \partial x^2} + \frac{12 + 10\nu}{5}\left(\frac{\rho I}{A}\right)^2\frac{\partial^4 v_0}{\partial t^4} = 0. \tag{23}$$

Equations (22) and (23) differ only in the coefficient of the fourth term by $1/5\nu(\rho I/A)^2$. The relative difference between these terms is 2% for $\nu = 0.3$. This suggests that for vibration problems, using the present theory with Cowper's shear correction factor and a body-force correction is essentially equivalent to using Timoshenko's shear correction factor and the equations of motion he originally derived. This agreement should be expected as experiments based on the natural frequencies of vibration have typically demonstrated that Timoshenko's shear correction factor is superior (Spence and Seldin, 1970; Kaneko, 1975).

6. Results

In this section, comparisons of the above formula are made with two cases: a three-layer symmetric beam, and a multi-layer beam composed of alternating materials. Results from a finite-element analysis are used to compare with the formulas derived above.

The first beam considered is composed of three layers, where the middle layer is made of a material that has reduced Young's modulus compared with the outer layers. This problem is designed to model a sandwich structure in which the inner core material is less stiff than the outer material. The outer two layers have Young's modulus E and Poisson's ratio ν while the inner core has Young's modulus αE and Poisson ratio ν. The depth of the beam is $2c$ and the extent of the inner core is from $y = -rc$ to $y = rc$, where r is the fraction of the beam that is composed of the core material. For this beam, simplifications from the general formulas above are possible. The average shear stiffness from Equation (14) simplifies to,

$$D_{33} = \frac{3EI}{2(1+\nu)(2c^3 - 3c^3 r(1-s))}, \tag{24}$$

where $s = (1-(1-\alpha)r^2)/\alpha$ and the shear correction factor from Equation (15) simplifies to,

$$k_{xy} = \frac{(1+\nu)(30r(s-1)+20)}{30(1+\nu)s - (6+8\nu) + 15(1+\nu)(1-s)(2+r^3-3r)}. \tag{25}$$

As the ratio of the Young's modulus of the core decreases, it is interesting to note that a limiting case is reached that is independent of the Poisson's ratio. This limit as $\alpha \to 0$ is,

$$k_{xy} = \frac{2}{3-r^2}. \tag{26}$$

The second beam considered is composed of alternating isotropic layers that have relative Young's modulus $E_1/E_2 = 10$ and Poisson's ratios of $\nu_1 = 0.2$ and $\nu_2 = 0.4$. For this case, we vary the number of layers, keeping the depth of the beam constant, $c = 1/2$ while varying the thickness of the layers to match. As a result $h_k = -c + 2c(k-1)/K$. The plies are composed of alternating material starting from the bottom layer. The beam is symmetric for odd K.

For finite-element calculations, we use bi-cubic Lagrange plane stress elements with a standard formulation. We choose these high-order elements because they capture the piecewise parabolic shear stress accurately through the thickness of the beam.

The finite-element model is constructed with $L/2c = 10$ with 50 elements along the length of the beam. For the three-layer beam, we take 20 elements through the thickness resulting in 18422 degrees of freedom. For the multi-layer beam the number of through-thickness elements varies so that the number of elements in each layer is the same while the total number of elements through the thickness does not fall below 20. Mathematically the number of elements through the thickness is $K\lceil 20/K \rceil$.

In order to compare the results with the corrections computed above, we compute the solution to a beam subject to a shear load at the tip, with the root fully fixed. We calculate the shear correction factor using Equation (9) at each Gauss point along the x-direction and integrate at these stations in the y-direction. Numerical integration

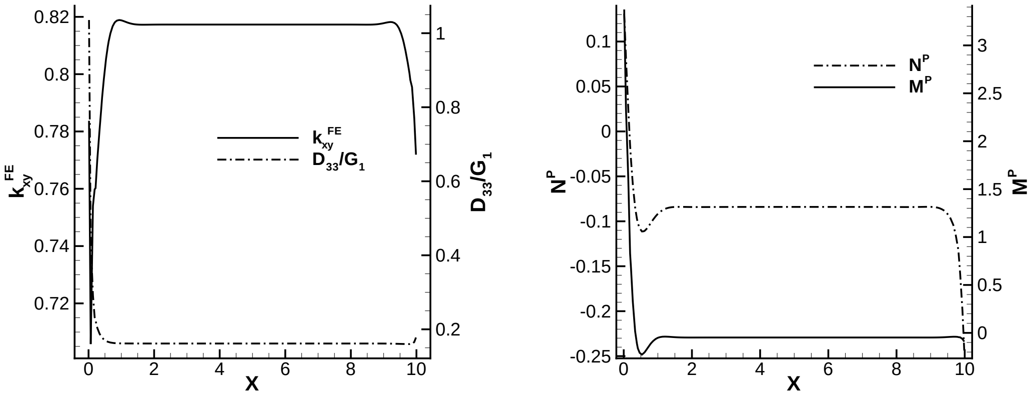

Figure 2: The variation of the computed shear correction factor, k_{xy}, homogenized shear stiffness D_{33} and the pressure corrections N^P and M^P per unit length of the multi-layer beam with $K = 5$. These results clearly show the end effects.

is used to evaluate the terms u_0, u_1 and v_0, and the derivative is performed using a central-difference calculation with $\Delta x = 10^{-5}$.

For comparison with the pressure corrections, we calculate a solution of a cantilevered beam subject to a pressure load distributed on the top and bottom with $P_b = 1/5$ and $P_t = 4/5$. The pressure correction is evaluated using a combination of finite-element and beam theory values where the total strain and stress moments are computed from the finite-element method, while the constitutive relation is used from Equation (7). This gives the following equation for the pressure correction to the axial-resultant,

$$N_{FE}^P = 2cD_{11} \left.\frac{\partial u_0}{\partial x}\right|_{FE} + \frac{2c^3}{3} D_{12} \left.\frac{\partial u_1}{\partial x}\right|_{FE} - N_{FE},$$

Similar expressions are used for the bending and shear corrections.

Typical results for the variation of shear correction factor, shear stiffness and pressure corrections with axial direction are plotted in Figure (2) for the multi-layer beam with $K = 5$. These show that there is a strong variation of these parameters close to the ends of the beam but that these quickly settle to a constant value over most of the length of the beam. For all comparisons in the following results, we average the shear correction factor, the shear stiffness and the pressure corrections obtained from the finite-element method over the span $x = 4$ to $x = 6$.

Figure (3) shows the variation of the shear correction factor and the average shear stiffness computed using Equation (25) and Equation (24) respectively. The finite-element calculations were performed at a core ratios of $r = 0.2, 0.5, 0.8, 0.9, 0.95, 0.98$ and at relative stiffness ratios of $\alpha = 1, 0.5, 0.1, 0.01$. Good agreement is obtained at all values. As well, Figure (3) shows the limiting case from Equation (26) for zero core stiffness.

Figure (4) shows the variation of the shear correction factor computed using the general form from Equation (15) and the homogenized shear stiffness computed using

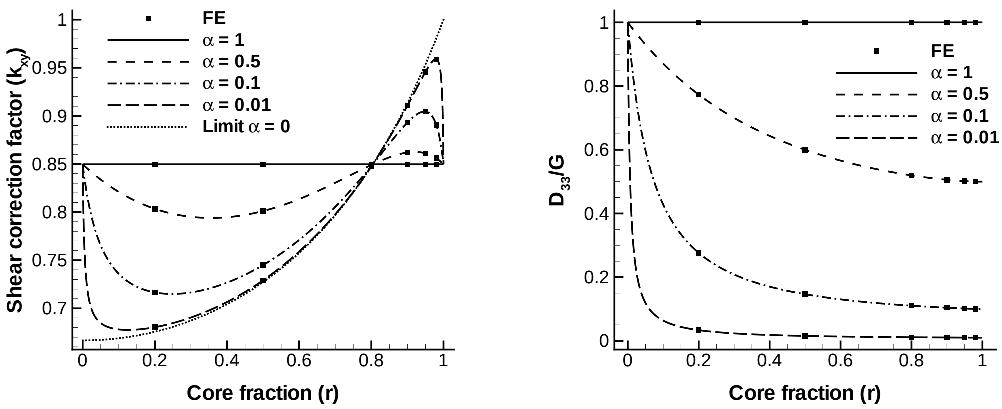

Figure 3: A comparison between the shear correction factor k_{xy} and homogenized shear stiffness D_{33} computed using the theory and the finite-element method for the three-layer beam.

Equation (14) for the multi-layer beam. Finite-element calculations are performed for the first 10 beams with $K = 1, \dots, 10$, while the analytic formulas are used up to $K = 50$ to show the trend. As was mentioned previously, for odd K, the beams are symmetric and for even K the beams exhibit bending-stretching coupling. As K becomes larger, the coefficients tend towards a limiting case. Excellent agreement is obtained. The average relative error for $K = 1, \dots, 10$ is 3.3×10^{-6} and 1.7×10^{-5} for the shear strain correction and shear stiffness respectively, while the maximum errors are 1.0×10^{-5} and 9.3×10^{-5} respectively.

Figure (5) shows the pressure corrections for the axial resultant and bending moment for the multi-layer beam. The theoretical results were computed by first finding the strain moment corrections from Equation (17a) and Equation (17b) and multiplying by the average constitutive relation from Equation (7). The average relative error for the pressure corrections are, 3.6×10^{-5} and 4.6×10^{-5} for N^P and M^P respectively while the maximum errors are, 1.8×10^{-4} and 1.2×10^{-4}. These results demonstrate that the constitutive equation is modified by the presence of an externally applied pressure load otherwise the predicted correction would be zero. In addition, these results show that these corrections are correctly predicted by Equation (17).

7. Conclusions

A Timoshenko beam theory for plane stress problems has been presented. Following the work of Prescott (1942) and Cowper (1966), the beam kinematics are developed in terms of average through-thickness displacement and rotation variables. The proposed theory includes a consistent method for calculating the stiffness of the beam, the shear strain correction factor and the strain-moment corrections for externally applied loads. These values are based on the axially-invariant fundamental state solutions. We have demonstrated that the present approach easily handles layered beam constructions

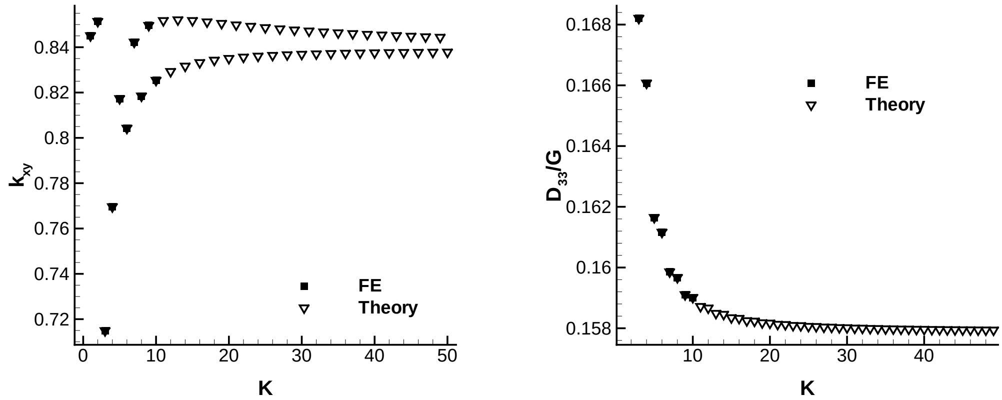

Figure 4: A comparison between the shear correction factor k_{xy} and homogenized shear stiffness D_{33} computed by theory and the finite-element method for the multi-layer beam.

through the use of both stress and strain moments that admit solutions where components of stress and strain may be discontinuous across interfaces. The external load corrections proposed in the theory modify the constitutive relationship and the equations of motion. The analysis presented suggests that for vibration problems using the present theory with Cowper's shear correction factor and a body-force correction is essentially equivalent to using Timoshenko's shear correction factor with the original equations of motion he derived. On the other hand, numerical comparisons using static analysis demonstrated the accuracy and the consistency of the definitions of the shear strain correction factor and the external load corrections. Both static and dynamic situations are treated by the theory without inconsistency as a result of the external load correction terms.

Acknowledgments

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References

Cowper, G., 1966. The shear coefficient in Timoshenko's beam theory. Journal of Applied Mechanics 33 (5), 335–340.

Dong, S., Alpdogan, C., Taciroglu, E., 2010. Much ado about shear correction factors in Timoshenko beam theory. International Journal of Solids and Structures 47 (13), 1651 – 1665.

Flower, H., Soutis, C., 2003. Materials for airframes. The Aeronautical Journal 107 (1072), 331–341.

Guiamatsia, I., 2010. A new approach to plate theory based on through-thickness homogenization. International Journal for Numerical Methods in Engineering.

Guiamatsia, I., Hansen, J. S., 2004. A homogenization-based laminated plate theory. ASME Conference Proceedings 2004 (47004), 421–436.

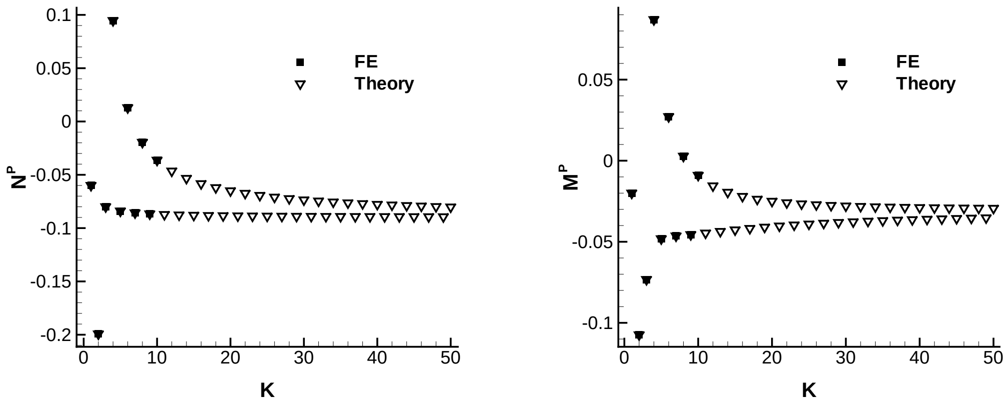

Figure 5: A comparison between the pressure corrections N^P and M^P computed using the theory $K = 1 \ldots 50$ and the finite-element method $K = 1 \ldots 10$.

Hansen, J. S., Almeida, S. d., April 2001. A theory for laminated composite beams. Tech. rep., Instituto Tecnológico de Aeronáutica.

Hansen, J. S., Kennedy, G. J., de Almeida, S. F. M., 2005. A homogenization-based theory for laminated and sandwich beams. In: 7th International Conference on Sandwich Structures. Aalborg University, Aalborg, Denmark, pp. 221–230.

Hutchinson, J. R., 1980. Vibration of solid cylinders. Journal of Applied Mechanics 47 (12), 901–907.

Hutchinson, J. R., 1981. Transverse vibration of beams, exact versus approximate solutions. Journal of Applied Mechanics 48 (12), 923–928.

Hutchinson, J. R., 2001. Shear coefficients for Timoshenko beam theory. Journal of Applied Mechanics 68 (1), 87–92.

Kaneko, T., 1975. On Timoshenko's correction for shear in vibrating beams. Journal of Physics D: Applied Physics 8 (16), 1927–1936.

Leibowitz, R. C., Kennard, E. H., 1961. Theory of freely-vibrating nonuniform beams, including methods of solution and application to ships.

Pickett, G., 1944. Application of the Fourier method to the solution of certain boundary problems in the theory of elasticity. Journal of Applied Mechanics 11, 176–182.

Prescott, J., 1942. Elastic waves and vibrations of thin rods. Philosophical Magazine 33, 703–754.

Shames, I. H., Dym, C. L., 1985. Energy and Finite Element Methods in Structural Mechanics. McGraw Hill Higher Education.

Spence, G. B., Seldin, E. J., 1970. Sonic resonances of a bar and compound torsion oscillator. Journal of Applied Physics 41 (8), 3383–3389.

Stephen, N. G., 1980. Timoshenko's shear coefficient from a beam subjected to gravity loading. Journal of Applied Mechanics 47 (1), 121–127.

Stephen, N. G., 2001. Discussion: "Shear coefficients for Timoshenko beam theory". Journal of Applied Mechanics 68 (11), 959–960.

Stephen, N. G., Levinson, M., 1979. A second order beam theory. Journal of Sound and Vibration 67 (3), 293–305.

Tafeuvoukeng, I. G., 2007. A unified theory for laminated plates. Ph.D. thesis, University of Toronto.

Timoshenko, S. P., 1921. On the correction for shear of the differential equation for transverse vibrations of prismatic bars. Philosophical Magazine 41, 744–746.

Timoshenko, S. P., 1922. On the transverse vibrations of bars of uniform cross-section. Philosophical Magazine 43, 125 – 131.